Nonlinear Schrödinger equations with mean terms in nonresonant multidimensional quadratic materials

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We derive the asymptotic equations governing the evolution of a quasi-monochromatic optical pulse in a nonresonant quadratic material starting from Maxwell equations. Under rather general assumptions, equations of nonlinear Schrödinger (NLS) type with coupling to mean fields result (here called NLSM). In particular, if the incident pulse is polarized along one of the principal axes of the material, scalar NLSM equations are obtained. For a generic input, however, coupled vector NLSM systems result. Special reductions of these equations include the usual scalar and vector NLS equations. Based on results known for similar systems which arise in other physical contexts, we expect the behavior of the solutions to be characterized by a rather large variety of phenomena. In particular, we show that the presence of the coupling to the dc fields can have a dramatic effect on the dynamics of the optical pulse, and stable localized multidimensional pulses can arise through interaction with boundary terms associated to the mean fields.

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I. INTRODUCTION

The propagation of a light pulse in a nonlinear medium is characterized by a series of complex changes in its structure. A theoretical interpretation of these phenomena is crucial in order to understand the nonlinear properties of the medium and the complex dynamics of such nonlinear systems. Quadratically nonlinear media are potentially promising for a number of applications. Propagation of a single light beam in a quadratic medium has attracted significant interest in recent years (see, e.g., Refs. [1–3] and references therein). In this kind of media the nonlinear response generates second harmonic components and dc fields which subsequently play a key role in the evolution of the optical pulse. Here we concentrate on novel system of equations which arise due to the interaction between the fundamental and dc fields in the situation where second-harmonic generation is not phase matched. In this case, the second harmonic component can be solved explicitly and produces an additional self-phase modulation contribution due to cascaded nonlinearity. The importance of the coupling between dc fields and the fundamental was realized early in nonlinear optics—in a different context [4,5]. The interaction between optical pulses and dc fields was also recently studied in Refs. [1,6,7].

In more familiar isotropic (Kerr) media, where the nonlinear response of the material depends cubically on the applied field, the dynamics of a quasi-monochromatic optical pulse is governed by the nonlinear Schrödinger equation (hereafter NLS). The NLS equation (first derived in optics in 1965 [8,9] and in a general context in 1967 [10]) is also a centrally important equation in other areas like fluid dynamics, plasma physics, etc. In optics, its spatial and temporal versions provide a framework to describe and explain a large variety of phenomena, from optical switching to long-distance communication systems. In one-dimensional nonresonant quadratic materials the NLS equation was recently derived as the evolution equation of a single optical pulse in Refs. [2,3]. The evolution of two pulses at the fundamental and second harmonic in a nonresonant one-dimensional material with both quadratic and cubic response was studied in Ref. [11], where coupled equations of NLS-type were shown to appear for the two wave components.

The above cited studies are relative to the one-dimensional temporal case, in which the modulation of the pulse envelope over the transverse coordinates is neglected. However, it is well known that $(1+1)$-dimensional structures propagating in a multidimensional medium are unstable with respect to modulations along the direction perpendicular to the structure (this instability is in a sense the analog in several dimensions of the well-known Benjamin-Feir instability [12] in one-dimensional systems). As a consequence, in a medium which has large transverse extent it is not desirable to reduce the dynamics of the pulse to a simple one-dimensional system. When studying the modulation of a wave packet in a multidimensional dispersive medium, generalized NLS systems with coupling to a mean term are known to appear in various physical situations [13,14]. Hereafter we denote such equations as NLSM. In some special cases these systems are also known to be integrable; the limiting integrable case was first studied in Ref. [15] in the context of water waves, and has been the subject of many research papers ever since (cf. Refs. [16–18]). However,

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even in the general, nonintegrable case these systems exhibit interesting phenomena depending on the coefficients in the equations, such as focusing, singularities and a rich structure of solutions (see, e.g., Refs. [19,20]).

In a recent letter [21] we discussed how novel NLSM systems appear in the context of nonlinear optics when studying three-dimensional bulk materials with nonzero $\chi^{(2)}$, i.e., with quadratic nonlinearity. A brief discussion of this subject also appears in Ref. [22], where a prototypical two-dimensional scalar system is derived from a model scalar wave equation. However, as shown in Ref. [21] and as we further explain here, the results obtained from the vector Maxwell equations in the full three-dimensional case differ significantly.

The structure of this work is the following. In Sec. II we present in detail the general framework that we use to derive the asymptotic equations of these systems. In Sec. III we provide a full derivation of the scalar NLSM system presented in Ref. [21], and we discuss in some detail the results arising in different physical situations. In Sec. IV we show how the same type of derivation can be extended to obtain novel systems of vectors, coupled NLSM equations, and discuss some of the different physical scenarios that can arise. Finally, in Sec. V, we numerically obtain localized solutions of the optical NLSM equations, and we show that the dynamics of the optical pulse can be significantly affected by the presence of the associated dc fields.

These results provide a generalization in quadratic materials of the well-known scalar and vector nonlinear Schrödinger equations—to which they reduce in some appropriate limits. Since both the scalar and vector versions of the nonlinear Schrödinger equation have proven to be of fundamental importance in many different aspects of nonlinear optics, we can expect our results to be of general character and to have a wide range of applicability.

II. THE PERTURBATION SCHEME

In this section we present the general framework that we will use in Secs. III and IV to derive the asymptotic equations of the system from the full $(3+1)$-dimensional Maxwell equations.

A. The vector nonlinear wave equation

In nonmagnetic materials, and in the absence of sources, Maxwell equations yield the vector nonlinear wave equation for the electric field $\mathbf{E}$ as

$$\nabla^2\mathbf{E}-\nabla(\nabla\cdot\mathbf{E})-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\mathbf{E}+\mathbf{P})=0, \tag{2.1}$$

where c is the speed of light in vacuum. The material polarization $\mathbf{P}$ is expressed in terms of the electric field by the expansion

$$\mathbf{P}=\chi^{(1)}_{\cdot\cdot}*\mathbf{E}+\chi^{(2)}_{\cdot\cdot\cdot}*\mathbf{EE}+\chi^{(3)}_{\cdot\cdot\cdot\cdot}*\mathbf{EEE}+\cdots, \tag{2.2}$$

where $\chi^{(n)}_{\cdot\cdot\cdots\cdot}(t_1,\ldots,t_n)$ is the nth order susceptibility of the material, and the asterisk denotes a n-dimensional convolution integral. Although not independent from the nonlinear wave equation (2.1), it is advantageous to use the divergence law:

$$\nabla\cdot(\mathbf{E}+\mathbf{P})=0. \tag{2.3}$$

We note that, as a consequence of equation (2.3), $\nabla\cdot\mathbf{E}=-\nabla\cdot\mathbf{P}\neq 0$. Since the term $\nabla\cdot\mathbf{E}$ is usually a small perturbation, its contribution to the nonlinear wave equation (2.1) is often neglected. However, its presence is sometimes crucial in the derivation of the correct asymptotic equations of the system, as we are going to show.

B. The multiple scale expansion

We study the evolution of a quasi-monochromatic wave with central frequency ω. We consider an incident field propagating along the z axis and polarized along one or both the transverse axes, where our coordinate axes are taken to be an appropriate permutation of the crystallographic axes of the material (see following sections). Our derivation is based on a multiple scale perturbation expansion. We define the rapidly varying phase

$$\theta=kz-\omega t, \tag{2.4a}$$

and slowly varying space and time coordinates as

$$X=\epsilon x,\quad Y=\epsilon y,\quad Z=\epsilon^2 z,\quad T=\epsilon(t-z/v), \tag{2.4b}$$

where $k(\omega)$ and $v(\omega)$ are, respectively, the wave number and the group velocity, to be determined later, and ϵ is a parameter that measures the width of the frequency spectrum of the input pulse: that is, $\epsilon\sim(\Delta\omega)/\omega$. Concretely speaking, we use a slowly varying envelope approximation, i.e., we assume that the modulation of the amplitude of the electromagnetic field occurs over scales which are much longer than the optical wavelength. This assumption motivates the substitutions

$$\frac{\partial}{\partial x}=\epsilon\frac{\partial}{\partial X}, \tag{2.5a}$$

$$\frac{\partial}{\partial y}=\epsilon\frac{\partial}{\partial Y}, \tag{2.5b}$$

$$\frac{\partial}{\partial z}=k\frac{\partial}{\partial\theta}-\frac{\epsilon}{v}\frac{\partial}{\partial T}+\epsilon^2\frac{\partial}{\partial Z}, \tag{2.5c}$$

$$\frac{\partial}{\partial t}=-\omega\frac{\partial}{\partial\theta}+\epsilon\frac{\partial}{\partial T}. \tag{2.5d}$$

C. Perturbation expansion of the optical field

We expand the electric field in powers of ϵ. Given the field at $O(\epsilon)$ one can deduce the higher order terms in the usual manner for such perturbation expansions. Explicitly, we write each Cartesian component j $(j=x,y,z)$ of the electric field as

$$E_j=\epsilon E_j^{(1)}+\epsilon^2 E_j^{(2)}+\epsilon^3 E_j^{(3)}+\cdots. \tag{2.6}$$

At any order in ϵ, $E_j^{(n)}$ is found to consist only of a finite number of higher harmonics generated by the nonlinear polarization $\mathbf{P}$. As a result, at any order in ϵ we can decompose the electric field into a sum of harmonic frequencies, each modulated by a complex envelope which is assumed to be slowly varying:

$$E_j^{(n)}=\sum_{m=-n}^{n} e^{im\theta}E_{j,m}^{(n)}(X,Y,Z,T). \tag{2.7}$$

The assumption of a quasi-monochromatic input field guarantees that each of the envelopes $E_{j,m}^{(n)}$ is centered around zero frequency. (As a consequence, the term $e^{im\theta}E_{j,m}^{(n)}$ is centered around the frequency $m\omega$.) Also, due to the reality of the electric field, $E_{j,m}^{(n)}=(E_{j,-m}^{(n)})^*$, where the asterisk denotes the complex conjugate.

D. Perturbation expansion of the material polarization

The polarization vector $\mathbf{P}$ can also be expanded in powers of ϵ. Substituting the electric field in Eq. (2.2) we employ relations such as

$$\int_{-\infty}^{\infty}\chi_{ab}^{(1)}(t-t')E_{b,m}^{(n)}(X,Y,Z,T')e^{im\theta'}\,dt' =\hat{\chi}_{ab}^{(1)}\left(\omega_m+i\epsilon\frac{\partial}{\partial T}\right)E_{b,m}^{(n)}(X,Y,Z,T)e^{im\theta}, \tag{2.8}$$

where for convenience we define $\omega_m=m\omega$, with

$$\hat{\chi}_{ab}^{(1)}\left(\overline{\omega}+i\epsilon\frac{\partial}{\partial T}\right)=\hat{\chi}_{ab}^{(1)}(\overline{\omega})+i\epsilon\frac{\partial}{\partial\omega}\hat{\chi}_{ab}^{(1)}(\omega)|_{\overline{\omega}}\frac{\partial}{\partial T} -\frac{1}{2}\epsilon^2\frac{\partial^2}{\partial\omega^2}\hat{\chi}_{ab}^{(1)}(\omega)|_{\overline{\omega}}\frac{\partial^2}{\partial T^2}+\cdots \tag{2.9}$$

and where $\hat{\chi}^{(1)}(\omega)$ is the Fourier transform of $\chi^{(1)}(t)$. Similar relations apply for the $\chi^{(2)}$ and $\chi^{(3)}$ contributions when Eq. (2.9) is substituted by a multivariate Taylor expansion. Our task is to expand the vector wave equation in powers of ϵ and to solve iteratively for the quantities $E_{j,m}^{(n)}$. While the results obtained with this procedure are perfectly equivalent to those coming from a more traditional perturbation method (cf. Ref. [10]), the merits of the expansion (2.7) manifest in (i) the substitution of differential equations for the $E_{j;m}^{(n)}$ with algebraic relations; and (ii) the effective decomposition of the problem into small units, which allows an easier identification of the relevant contributions at any stage. As a result, a substantial simplification of the calculations is obtained. This will be particularly useful when deriving the vector NLSM equations, where the number of different nontrivial contributions in the expansion is considerably large.

E. Perturbation expansion of the wave equation

Motivated by the previous considerations we substitute the expansions (2.6)–(2.7) into Eq. (2.1), which allows us to formally decompose the wave equation into a set of equations for the corresponding harmonic components:

$$L_{j,a;m}E_{a;m}-D_{j;m}+M_{j,a,b;r,m-r}E_{a;r}E_{b;m-r} +N_{j,a,b,c;r,s,m-r-s}E_{a;r}E_{b;s}E_{c;m-r-s}=0, \tag{2.10}$$

where a sum is understood over the Cartesian indices *a,b,c* and over the harmonic indices *r,s*. A similar decomposition holds for the divergence equation (2.3). The term D in (2.10) represents the gradient of the divergence of the electric field [i.e., $\mathbf{D}=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E})$], while L accounts for the Laplacian and the linear part of the material polarization, and *M,N* are the result of the quadratic and cubic nonlinearity of the system, respectively. Note that $\boldsymbol{\nabla}\cdot\mathbf{E}$ is at least $O(\epsilon^2)$, which allows us to treat D as a perturbation.

By using Eqs. (2.6)–(2.7) we have $L_{j,a;m}=L_{j;m}\delta_{j,a}$, where $\delta_{i,j}$ is Kronecker's delta,

$$L_{j;m}=\left(imk-\frac{\epsilon}{v}\partial_T+\epsilon^2\partial_Z\right)^2+\epsilon^2(\partial_X^2+\partial_Y^2)+\kappa_j^2(\omega_m+i\epsilon\partial_T), \tag{2.11}$$

with $\kappa_j(\omega)$ defined by

$$\kappa_j^2(\omega)=\frac{\omega^2}{c^2}[1+\hat{\chi}_{jj}^{(1)}(\omega)], \tag{2.12}$$

and where only materials for which the principal and crystallographic coordinate systems coincide are considered. The operators M and N are

$$M_{j,a,b;r,m-r}=\frac{1}{c^2}[\omega_m+i\epsilon(\partial_{T_a}+\partial_{T_b})]^2\cdot\hat{\chi}_{j,a,b}^{(2)}(\omega_r +i\epsilon\partial_{T_a},\omega_{m-r}+i\epsilon\partial_{T_b}), \tag{2.13a}$$

$$N_{j,a,b,c;r,s,m-r-s}=\frac{1}{c^2}[\omega_m+i\epsilon(\partial_{T_a}+\partial_{T_b}+\partial_{T_c})]^2\hat{\chi}_{j,a,b,c}^{(3)}(\omega_\tau +i\epsilon\partial_{T_a},\omega_s+i\epsilon\partial_{T_b},\omega_{m-r-s}+i\epsilon\partial_{T_c}), \tag{2.13b}$$

where $\partial_{T_a}(E_{a;r}E_{b;m-r})=(\partial_T E_{a;r})E_{b;m-r}$, etc. As a result of the combined expansions (2.6)–(2.7), *L,M,N* are also automatically expanded in powers of ϵ. That is,

$$L_m=L_m^{(0)}+\epsilon L_m^{(1)}+\epsilon^2 L_m^{(2)}+\cdots, \tag{2.14}$$

and similar for M and N. It is useful to list the first few terms in these expansions:

$$L_{j;m}^{(0)}=-(mk)^2+\kappa_j^2(\omega_m), \tag{2.15a}$$

$$L_{j;m}^{(1)}=2i[-mk/v+(\kappa_j\kappa_j')\omega_m]\partial_T, \tag{2.15b}$$

$$L_{j;m}^{(2)} = 2imk\partial_Z + \partial_{XX} + \partial_{YY} - [(\kappa_j\kappa_j'' + (\kappa_j')^2)_{\omega_m} - 1/v^2]\partial_{TT}. \tag{2.15c}$$

Also,

$$M_{s,m-s}^{(0)} = \frac{\omega_m^2}{c^2}\hat{\chi}_{s,m-s}^{(2)}, \tag{2.16a}$$

$$M_{s,m-s}^{(1)} = i\frac{\omega_m^2}{c^2}(\hat{\chi}_{s',m-s}^{(2)}\partial_{T_a} + \hat{\chi}_{s,m-s'}^{(2)}\partial_{T_b}) + 2i\frac{\omega_m}{c^2}\hat{\chi}_{s,m-s}^{(2)}(\partial_{T_a} + \partial_{T_b}), \tag{2.16b}$$

$$M_{s,m-s}^{(2)} = -\frac{\omega_m^2}{2c^2}(\hat{\chi}_{s'',m-s}^{(2)}\partial_{T_aT_a} + 2\hat{\chi}_{s',m-s'}^{(2)}\partial_{T_aT_b} + \hat{\chi}_{s,m-s'}^{(2)}\partial_{T_bT_b}) \tag{2.16c}$$

$$-2\frac{\omega_m}{c^2}(\partial_{T_a} + \partial_{T_b})(\hat{\chi}_{s',m-s}^{(2)}\partial_{T_a} + \hat{\chi}_{s,m-s'}^{(2)}\partial_{T_b}) - \frac{1}{c^2}\hat{\chi}_{s,m-s}^{(2)}(\partial_{T_a} + \partial_{T_b})^2, \tag{2.16d}$$

where $\hat{\chi}_{s,m-s}^{(2)} = \hat{\chi}^{(2)}(\omega_s, \omega_{m-s})$, and where for simplicity tensor indices have been dropped. Similarly

$$N_{r,s,m-s}^{(0)} = \frac{\omega_m^2}{c^2}\hat{\chi}_{r,s,m-r-s}^{(3)}. \tag{2.17}$$

At any order n and in each direction j, the product $L_{j;m}E_{j;m}$ decomposes as

$$L_mE_m = L_m^{(0)}E_m^{(n)} + L_m^{(1)}E_m^{(n-1)} + \cdots + L_m^{(n-1)}E_m^{(1)}, \tag{2.18}$$

where again the index j has been omitted for simplicity. A similar decomposition holds for the nonlinear terms and for the divergence term $D_{j;m}$ in Eq. (2.10).

In the next sections, using these expansions, we solve Eq. (2.10) and the divergence Eq. (2.3) recursively in powers of ϵ in a number of representative cases.

III. SCALAR NLSM EQUATIONS

In this section we consider the case in which the electric field is polarized along one of the principal axes of the material. (That is, the incident field has a nonzero component only along one of the principal axes.) In this case only a relatively small number of terms in the vector wave equation (2.1) plays a significant role, and we are able to obtain a relatively simple scalar result. To be specific, we consider the propagation of a light pulse in a (uniaxial) tetragonal 4 mm-material. Materials with such a symmetry class are provided, for instance, by photorefractive $BaTiO_3$, SBN, and KTN [23]. This particular choice of symmetry class is motivated by the special structure assumed by the nonlinear susceptibility tensors $\chi^{(2)},\chi^{(3)}$ (cf. Ref. [24]), and guarantees that only few components play an active role in the vector wave equation (2.1). For convenience of exposition, we take our x axis to coincide with the crystallographic z axis of the material.

The nonzero entries of the susceptibility tensors are the following.

(1) For $\chi^{(1)}, \chi_{xx}^{(1)}: \chi_{yy}^{(1)} = \chi_{zz}^{(1)}$.

(2) For $\chi^{(2)}$: In the x direction, $\chi_{xxx}^{(2)}$, $\chi_{xyy}^{(2)} = \chi_{xzz}^{(2)}$; in the y direction, $\chi_{yyx}^{(2)}, \chi_{yxy}^{(2)}$; in the z direction, $\chi_{zzx}^{(2)} = \chi_{yyx}^{(2)}$, $\chi_{zxz}^{(2)} = \chi_{yxy}^{(2)}$.

(3) For $\chi^{(3)}$: The three elements with all indices equal $\chi_{xxxx}^{(3)} = \chi_{yyyy}^{(3)}$; $\chi_{zzzz}^{(3)}$. Also, the 18 elements with indices equal in pairs, with equality between elements obtained by exchanging $x \leftrightarrow y$ (i.e., $\chi_{zxzx}^{(3)} = \chi_{zyzy}^{(3)}$, etc.). In total, this leaves 11 independent elements.

A. Derivation of the scalar equations

We take the leading order incident field to consist only of the fundamental polarized in the x direction. That is, we take $E_{x,m}^{(1)} = 0$ for $m \neq \pm 1$ and $E_{y,m}^{(1)} = E_{z,m}^{(1)} = 0$.

$O(\epsilon)$: From the x component ($j = x$) of Eq. (2.10) at the fundamental ($m = 1$) at $O(\epsilon)$ we obtain the usual dispersion relation: $k \equiv \kappa_x(\omega)$, with $\kappa_j(\omega)$ defined by (2.12). Here we neglect the imaginary part of $\hat{\chi}^{(1)}(\omega)$, which leads to attenuation. The effects of loss can be included in the theory in a straightforward way. Note that, due to the uniaxiality, $\chi_{yy} = \chi_{zz}$ (i.e., $\chi_{xx} = \chi_{yy}$ in the crystallographic system), which will be important later.

$O(\epsilon^2)$: For each Cartesian component j, the relevant combination for the linear part of the wave equation (2.10) is $L_{j;m}^{(1)}E_{j;m}^{(1)} + L_{j;m}^{(0)}E_{j;m}^{(2)}$. Then, from the x component of the wave equation at the fundamental ($j = x$, $m = 1$), we obtain the group velocity $v \equiv v_x(\omega)$, with

$$v_j(\omega) = 1/\kappa_j'(\omega). \tag{3.1}$$

Also, at $m = 2$ we have $E_{y,2}^{(2)} = E_{z,2}^{(2)} = 0$, from the y and z components, and from the x component we find the second harmonic generated by the quadratic nonlinearity:

$$E_{x,2}^{(2)} = \frac{\hat{\chi}_{xxx}^{(2)}(\omega,\omega)}{\Delta_x^2(\omega)}(E_{x,1}^{(1)})^2, \tag{3.2}$$

where

$$\Delta_j^2(\omega) = n_j^2(\omega) - n_j^2(2\omega). \tag{3.3}$$

We emphasize that in our derivation we assume $\Delta_x^2(\omega) \gg \epsilon$; i.e., we assume to be far away from the phase matching condition which leads to second harmonic resonance. We also note that the particular choice of material (reflecting in the tensor structure of $\chi^{(2)}$) ensures that no new Cartesian components are generated as an effect of the nonlinearity. At $O(\epsilon^2)$, all the other harmonics are found to be zero with one important exception for $m = 0$: Due to the absence of fast derivatives, the dc electric field (i.e., $m = 0$) is undetermined at this stage; that is, $E_{x,0}^{(2)} \neq 0$. In fact, at $O(\epsilon^2)$, similar dc fields are also allowed for the y and z components. Like the field at the fundamental $E_{x,1}^{(1)}$, the dc fields $E_{x,0}^{(2)}$, $E_{y,0}^{(2)}$, and $E_{z,0}^{(2)}$ need to be determined at higher orders in the expansion. As it turns out, all dc fields play a crucial role in the calculation. It is important to realize that such mean terms cannot simply be ignored, otherwise inconsistencies develop at higher orders in the expansion, as we are going to show. It is also important to observe that, at this stage, the results of the derivation would be exactly the same for Kerr materials (with the exception that, for Kerr materials, $\chi^{(2)}=0$ and no second harmonic is produced). The difference lies on the fact that, in the absence of quadratic nonlinearity, at higher orders there would be no source term in the equations for the dc fields and no coupling to the dc fields in the equations for the optical fields.

Finally, at $O(\epsilon^2)$, the divergence equation (2.3) at the fundamental determines the z component of the optical field, which is generated by the slow modulation of the x component:

$$E_{z,1}^{(2)}=\frac{in_x^2(\omega)}{\kappa_x(\omega)n_z^2(\omega)}\frac{\partial}{\partial X}E_{x,1}^{(1)}, \tag{3.4}$$

where the linear index of refraction is defined in the usual way as

$$n_j^2(\omega)=1+\hat{\chi}_{jj}^{(1)}(\omega). \tag{3.5}$$

The fast variation of $\partial_X E_{x,1}^{(1)}$ with respect to θ generates a nonzero z component at order $O(\epsilon^2)$. However, since the modulation of the envelope with respect to y is slow, no such component is generated in the y direction. That is, $E_{y,1}^{(2)}=0$.

$O(\epsilon^3)$: The divergence law (2.3) at dc ($m=0$) yields an explicit equation for $E_{z,0}^{(2)}$ in terms of the fundamental $E_{x,1}^{(1)}$ and the dc fields in the x and y directions, $E_{x,0}^{(2)}$ and $E_{y,0}^{(2)}$:

$$\frac{\partial}{\partial T}E_{z,0}^{(2)}=\frac{v(\omega)}{n_z^2(0)}\bigg[n_x^2(0)\frac{\partial}{\partial X}E_{x,0}^{(2)}+n_y^2(0)\frac{\partial}{\partial Y}E_{y,0}^{(2)} +2\hat{\chi}_{xxx}^{(2)}(\omega,-\omega)\frac{\partial}{\partial X}(|E_{x,1}^{(1)}|^2)\bigg]. \tag{3.6}$$

It is noted that the presence of the source term proportional to $|E_{x,1}^{(1)}|^2$ makes it impossible to neglect the dc fields and maintain consistency in the derivation.

For each Cartesian component j, the relevant combination for the linear part of the wave equation (2.10) is $L_{j;m}^{(3)}E_{j;m}^{(1)}+L_{j;m}^{(1)}E_{j;m}^{(2)}+L_{j;m}^{(0)}E_{j;m}^{(3)}$. From the x component at the fundamental, we find the evolution of the slowly varying amplitude of the incident field. Defining

$$E_{x,1}^{(1)}=:A(X,Y,T,Z),\quad E_{x,0}^{(2)}=:\phi(X,Y,Z,T), \tag{3.7}$$

and using Eq. (3.2) in the nonlinear contributions and (3.6) to compute D_x (the divergence term), we get for A the following evolution equation:

$$\bigg[2ik\frac{\partial}{\partial Z}+(1-\alpha_{x,1})\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}-kk''\frac{\partial^2}{\partial T^2} +M_{x,1}|A|^2+M_{x,0}\phi\bigg]A=0, \tag{3.8a}$$

where the coefficients $\alpha_{x,1}$, $M_{x,0}$, and $M_{x,1}$ are given below. Equation (3.8a) contains the coupling of the fundamental with the second harmonic and the mean field ϕ. An explicit expression for ϕ cannot be obtained. Rather, an evolution equation for ϕ in terms of the fundamental is found. The two necessary ingredients for this kind of coupling to occur are the presence of a multidimensional medium and quadratic nonlinearity.

$O(\epsilon^4)$: The evolution of the mean field ϕ is captured from the wave equation at dc at $O(\epsilon^4)$, where, using (3.6), it is found that

$$\bigg[(1-\alpha_{x,0})\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}+s_x\frac{\partial^2}{\partial T^2}\bigg]\phi =\bigg[N_{x,1}\frac{\partial^2}{\partial T^2}-N_{x,2}\frac{\partial^2}{\partial X^2}\bigg](|A|^2). \tag{3.8b}$$

Once ϕ is known, the auxiliary field $E_{y,0}^{(2)}$ is obtained from ϕ as

$$\bigg[\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}+s_y\frac{\partial^2}{\partial T^2}\bigg]E_{y,0}^{(2)}-\alpha_{x,0}\frac{\partial^2}{\partial X\partial Y}\phi =-N_{x,2}\frac{\partial^2}{\partial X\partial Y}(|A|^2). \tag{3.9}$$

The coefficient $\alpha_{j,m}$ appearing in Eqs. (3.8) is defined as

$$\alpha_{j,m}=1-\frac{n_j^2(\omega_m)}{n_z^2(\omega_m)}, \tag{3.10a}$$

while the $M_{j,m}$ and $N_{j,m}$ are given by

$$M_{j,0}=2\frac{\omega^2}{c^2}\tilde{\chi}_{jjj}^{(2)}(\omega,0), \tag{3.10b}$$

$$M_{j,1}=\frac{\omega^2}{c^2}\bigg[3\hat{\chi}_{jjjj}^{(3)}(\omega,\omega,-\omega)+\frac{2}{\Delta_j^2(\omega)} \times\hat{\chi}_{jjj}^{(2)}(2\omega,-\omega)\hat{\chi}_{jjj}^{(2)}(\omega,\omega)\bigg], \tag{3.10c}$$

$$N_{j,1}=\frac{2}{c^2}\hat{\chi}_{jjj}^{(2)}(\omega,-\omega),$$

$$N_{j,2}=c_z^2(0)N_{j,1}=\frac{2}{n_z^2(0)}\hat{\chi}_{jjj}^{(2)}(\omega,-\omega), \tag{3.10d}$$

and

$$s_j = \frac{1}{v^2(\omega)} - \frac{1}{c_j^2(0)}, \tag{3.10e}$$

where $c_j^2(\omega) = c^2/n_j^2(\omega)$ is the phase velocity. We recall that, for convenience, our choice of axes is a permutation of the crystallographic set. Also, although we only need the coefficients for $j=x$ at this time, the corresponding coefficients for $j=y$ will also be needed when considering vector NLSM equations.

We note that $N_{j,2}$ and the $\alpha_{j,m}$ arise from the vector nature of the electric field [via the contribution of $\nabla(\nabla\cdot\mathbf{E})$]. That is, $N_{j,2} = \alpha_{j,m} = 0$ if $\nabla\cdot\mathbf{E} = 0$. Also, $M_{x,1}$ results from the combined effects of $\chi^{(3)}$ and two $\chi^{(2)}$ in cascade: the first term in $M_{x,1}$ is due to the self-interaction of the fundamental, while the second originates from the coupling between first and second harmonic. Similar physical problems are known to be capable of leading to large ''effective'' third order processes [25,26]. Again, the choice of the symmetry class is instrumental in simplifying the contribution arising from $\chi^{(3)}$. In fact, due also to the particular input considered (namely, the fact that $E^{(1)}_{y,\pm1} = E^{(1)}_{z,\pm1} = 0$), only the tensor components $\chi^{(2)}_{xxx}$ and $\chi^{(3)}_{xxxx}$ of the nonlinear susceptibilities play a role in the calculations.

B. Remarks

Equations (3.8) constitute the fundamental scalar system that governs the evolution of a multidimensional quasi-monochromatic pulse in a nonresonant material with quadratic nonlinearity. They are the $(3+1)$-dimensional analogue in optics of the $(2+1)$-dimensional equations arising in water waves [13,15].

As a result of the perturbation expansion, the electric field $\mathbf{E}(x,y,z,t)$ is decomposed as follows:

$$\begin{aligned} E_x(x,y,z,t) &= \epsilon[E^{(1)}_{x,1}(X,Y,Z,T)e^{i\theta} + E^{(1)}_{x,1}(X,Y,Z,T)^* e^{-i\theta}] \\ &\quad + \epsilon^2[E^{(2)}_{x,2}(X,Y,Z,T)e^{2i\theta} + E^{(2)}_{x,2} \\ &\quad \times (X,Y,Z,T)^* e^{-2i\theta} + E^{(2)}_{x,0}(X,Y,Z,T)] \\ &\quad + O(\epsilon^3), \end{aligned} \tag{3.11a}$$

$$E_y(x,y,z,t) = \epsilon^2 E^{(2)}_{y,0}(X,Y,Z,T) + O(\epsilon^3), \tag{3.11b}$$

$$\begin{aligned} E_z(x,y,z,t) &= \epsilon^2[E^{(1)}_{z,1}(X,Y,Z,T)e^{i\theta} + E^{(1)}_{z,1}(X,Y,Z,T)^* e^{-i\theta} \\ &\quad + E^{(2)}_{z,0}(X,Y,Z,T)] + O(\epsilon^3), \end{aligned} \tag{3.11c}$$

where $E_{x,2}$ is given by Eq. (3.2), $E_{z,1}$ is given by Eq. (3.4), $E_{z,0}$ is given by Eq. (3.6), $E_{y,0}$ is given by Eq. (3.9), and $E_{x,1}$ and $E_{x,0}$ are determined by the NLSM equations (3.8).

The absence of a Z derivative in Eqs. (3.8b) and (3.9) originates from the choice of using a reference frame that is moving with the group velocity of the optical pulse. In fact, an alternative but equivalent derivation can be done without performing the transformation to the comoving frame and introducing the multiple time and space scales

$$T = \epsilon t, \quad Z_1 = \epsilon z, \quad Z_2 = \epsilon^2 z, \quad \ldots. \tag{3.12}$$

In this case the resulting equations would be

$$\begin{aligned} &2ik\left(\frac{\partial}{\partial Z} + \frac{1}{v(\omega)}\frac{\partial}{\partial T}\right)A + \epsilon\Big[(1-\alpha_{x,1})\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} - kk''\frac{\partial^2}{\partial T^2} \\ &\qquad + M_{x,1}|A|^2 + M_{x,0}\phi\Big]A = 0, \end{aligned} \tag{3.13a}$$

$$\begin{aligned} &\left[(1-\alpha_{x,0})\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2} - \frac{1}{v^2(0)}\frac{\partial^2}{\partial T^2}\right]\phi \\ &\qquad = \left[N_{x,1}\frac{\partial^2}{\partial T^2} - N_{x,2}\frac{\partial^2}{\partial X^2}\right](|A|^2), \end{aligned} \tag{3.13b}$$

where now $Z \equiv Z_1$.

We remark upon the importance of the sign of s_j in Eqs. (3.8b)–(3.9): If $s_j > 0$, Eqs. (3.8b) and (3.9) are elliptic, whereas if $s_j < 0$ (and $\alpha_{j,0} < 1$), they are hyperbolic. Indeed, in the case of all the materials considered here $s_j < 0$, which has important ramifications (cf. Sec. V). Finally, we observe that the standard NLS equation can be considered as a special ''limiting'' case where $\chi^{(2)} = 0$, in which case we have $M_{x,0} = N_{x,1} = N_{x,2} = 0$, $E^{(2)}_{j,0} = \text{const}$, and $M_{x,1} = 3(\omega/c)^2\hat{\chi}^{(3)}_{xxxx} \times(\omega,\omega,-\omega)$. Then, if we further assume $\chi^{(1)}$ to be isotropic, $\alpha_{x,1} = 0$ and we obtain the usual scalar multidimensional NLS equation for isotropic materials as a reduction of Eqs. (3.8).

We also emphasize that the special structure of Eqs. (3.8) depends on the particular choice of symmetry class, which crucially reflects on the type of nonlinear couplings between the different harmonic components. In general, different symmetry classes lead to different types of evolution equations. As an example, we consider a (biaxial) orthorhombic mm^2 class. (A material with such a symmetry class is provided by $KNbO_3$ [23].) The derivation of the fundamental equations proceeds exactly in the same way as before, and equations similar to the system Eqs. (3.8) are found. However, due to biaxiality, we now have $\chi_{yy} \neq \chi_{zz}$ (i.e., $\chi_{xx} \neq \chi_{yy}$ in the crystallographic system). This implies that $\alpha_{y,0} \neq 0$, and $E^{(2)}_{y,0}$ enters in the equation for $E^{(2)}_{x,0}$. That is, the resulting equations for the fundamental $E^{(1)}_{x,1} =: A(X,Y,T,Z)$ and for the mean fields $E_{x,0} =: \phi_x(X,Y,Z,T)$ and $E_{y,0} =: \phi_y(X,Y,Z,T)$ now read

$$\begin{aligned} &\Big[2ik\frac{\partial}{\partial Z} + (1-\alpha_{x,1})\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} - kk''\frac{\partial^2}{\partial T^2} \\ &\qquad + M_{x,1}|A|^2 + M_{x,0}\phi_x\Big]A = 0 \end{aligned} \tag{3.14a}$$

and

$$\begin{aligned} &\left[(1-\alpha_{x,0})\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + s_x\frac{\partial^2}{\partial T^2}\right]\phi_x - \alpha_{y,0}\frac{\partial^2}{\partial X\partial Y}\phi_y \\ &\qquad = \left[N_{x,1}\frac{\partial^2}{\partial T^2} - N_{x,2}\frac{\partial^2}{\partial X^2}\right](|A|^2), \end{aligned} \tag{3.14b}$$

$$\left[\frac{\partial^2}{\partial X^2}+(1-\alpha_{y,0})\frac{\partial^2}{\partial Y^2}+s_y\frac{\partial^2}{\partial T^2}\right]\phi_y-\alpha_{x,0}\frac{\partial^2}{\partial X\partial y}\phi_x$$

$$=-N_{x,2}\frac{\partial^2}{\partial X\partial Y}(|A|^2). \tag{3.14c}$$

As a consequence, Eqs. (3.14a) and (3.14b) do not form a closed system anymore, and the full set of Eqs. (3.14) is now necessary to describe the behavior of the material. In this case neglecting the term $\nabla\cdot\mathbf{E}$ in the nonlinear wave equation (2.1)—which corresponds to letting $\alpha_{j,m}=0$—would result in a different systems of equations, which miss the direct coupling between the two dc fields. We also emphasize that, even in the relatively simpler situation in which Eqs. (3.8a) and (3.8b) are sufficient to completely determine the evolution of the pulse, the underlying dynamics of the system is still characterized by highly nontrivial dc interactions in all Cartesian components. It is also evident that the NLSM equations are rather general. Several comments are now in order.

(1) As mentioned before, the above equations are derived under the assumption that there are no resonant wave interactions; otherwise the governing equations and relevant scales would be very different—e.g., two/three wave interactions, which have already been the subject of many research papers (see, e.g., Refs. [27,28] and references therein).

(2) We do not introduce dc fields and/or second harmonic components at leading order because we are interested in the evolution of a modulated optical field and not in the interactions among different waves. This is a standard assumption in order to obtain NLS-type—and in this case NLSM-type—equations. Of course, other assumptions would lead to different evolution equations, e.g., long–short wave interactions (cf. Refs. [6,7,20,29]).

(3) Again, we emphasize that the mean fields are driven by the optical field and play a central role in the equations. Indeed, as we have seen, it is necessary to incorporate the dc field in the analysis at $O(\epsilon^2)$; otherwise inconsistencies arise in the expansion. This is true even in the one-dimensional temporal case (i.e., when the mean fields are independent of X and Y). In this case however the mean fields can be integrated explicitly; only then the equations reduce to the well-known NLS equation (cf. Refs. [2,3]). In particular, if $\chi^{(2)}=0$ (i.e., for the familiar Kerr materials) there are no source terms in the analogue of Eqs. (3.14b)–(3.14c) and the mean fields are zero.

(4) Finally, if the modulations of the incident field are so slow that it can be assimilated to a continuous wave, the scenario reduces to that described in Ref. [1], in which a static dc field produces a change in the refractive index through the electro-optic effect. However, if a temporally or spatially modulated pulse is considered, the full NLSM system of equations is necessary for a more accurate description of the physics, in which a traveling dc field is obtained, and for which phase matching can occur mediated by the optical group velocity and the dc field phase velocity. Previous studies on the coupling between optical fields and their low frequency counterparts show that a traveling dc field is produced by a modulated optical envelope [4]. Our results show that a spatially modulated envelope works just as well as a temporally modulated one.

It is also worth discussing some results that are known for similar systems of $(2+1)$-dimensional NLSM equations which arise in water waves (cf. Ref. [19]), since we expect that many of the issues will be also relevant in our context.

C. NLSM equations in water waves

In the context of water waves, the relevant problem for our purposes is the evolution of a small-amplitude, slowly modulated packet of surface waves on sufficiently deep water. If A is the dimensionless envelope of the wave packet, propagating in the x direction, and Φ is the dimensionless amplitude of the mean fluid flow, the dynamical equations for A and Φ take the form [13]

$$i\frac{\partial A}{\partial\tau}+c_1\frac{\partial^2 A}{\partial\xi^2}+c_2\frac{\partial^2 A}{\partial\eta^2}=\chi_1 A\frac{\partial\Phi}{\partial\xi}+\chi_2|A|^2A, \tag{3.15a}$$

$$\gamma\frac{\partial^2\Phi}{\partial\xi^2}+\frac{\partial^2\Phi}{\partial\eta^2}=-\beta\frac{\partial[|A|^2]}{\partial\xi}, \tag{3.15b}$$

where $\xi=\epsilon k(x-c_g t)$, $\eta=\epsilon ky$ and $\tau=\epsilon^2(gk)^{1/2}t$ are the dimensionless coordinates, (k,l) are the wave numbers in the (x,y) directions, $c_g=\partial\omega/\partial k$ is the group velocity, and g is the gravity acceleration. The coefficients c_1, c_2, χ_1, χ_2, γ, and β are suitable functions of k, c_g, the dispersion coefficients $\partial^2\omega/\partial k^2$ and $\partial^2\omega/\partial l^2$, the water depth h and the surface tension T.

Depending on the values of the dimensionless quantities kh and $\tilde{T}=(k^2+l^2)T/g$, several physical scenarios arise, as discussed in detail in Refs. [19,20]. Also, different reductions are possible in different physical limits.

(1) When derivatives with respect to y can be neglected (e.g., in a narrow canal), Eq. (3.15b) can be integrated immediately, and one recovers the familiar one-dimensional nonlinear Schrödinger equation, which is a completely integrable infinite-dimensional Hamiltonian system that can be solved by the inverse scattering transform (IST) [20].

(2) In the deep water limit, $kh\to\infty$, the coefficient β tends to zero. Thus, the mean flow Φ vanishes and Eqs. (3.15) reduce to the $(2+1)$-dimensional NLS equation:

$$i\frac{\partial A}{\partial\tau}+c_1^\infty\frac{\partial^2 A}{\partial\xi^2}+c_2^\infty\frac{\partial^2 A}{\partial\eta^2}=\chi_2^\infty|A|^2A. \tag{3.16}$$

Contrary to the one-dimensional case, this equation is not solvable by IST. Also, for various choices of parameters (sufficiently strong surface tension in sufficiently deep water) the solutions can blow up in finite time. One-dimensional solitons (i.e., NLS solitons) embedded in the two-dimensional equation are unstable to slow transverse perturbations.

(3) A different scenario arises in the opposite limit, that is shallow water, when $kh\to 0$ with $\epsilon\ll(kh)^2$. In this case, after rescaling, the equations can be written as

$$i\frac{\partial A}{\partial t}-\sigma\frac{\partial^2 A}{\partial x^2}+\frac{\partial^2 A}{\partial y^2}=A\frac{\partial \Phi}{\partial x}+\sigma|A|^2A, \qquad (3.17a)$$

$$\sigma\frac{\partial^2 \Phi}{\partial x^2}+\frac{\partial^2 \Phi}{\partial \xi^2}=-2\frac{\partial[|A|^2]}{\partial x}, \qquad (3.17b)$$

where $\sigma=\mathrm{sign}(\frac{1}{3}-\tilde{T})$. This system, usually called the Davey–Stewartson equations [15], is of IST type, and thus completely integrable. We should note that if a similar situation were also true for the optical NLSM equations found here, the corresponding equations would provide one of the first integrable multidimensional systems in nonlinear optics.

In this last case—that is, for the Davey–Stewartson system—several exact solutions are available. In particular, stable localized pulses, often called ''dromions,'' are known to exist which are driven by the appropriate mean fields. In Sec. V we further discuss these solutions of the integrable case, and we show that similar solutions exist even in the more general nonintegrable case described in this work.

IV. VECTOR NLSM EQUATIONS

We now direct our attention to the case in which the electric field has nonzero components along both the transverse principal axes of the material. This allows us to derive new systems of equations, whose analog is not known to exist in other physical contexts. The derivation of these coupled equations closely follows the method developed for the scalar case. For concreteness, we consider a (uniaxial) hexagonal $\bar{6}$ material, where now we take x,y,z to coincide with the crystallographic axes of the material. Materials with other uniaxial symmetry classes such as 3 m (like $LiNbO_3$ and $LiTaO_3$, cf. Ref. [23]) will result in similar vector NLSM equations. This amounts to assuming the following structure for the nonlinear susceptibility tensors [24]:

(1) For $\chi^{(1)}$, only diagonal elements are nonzero: $\chi^{(1)}_{xx}=\chi^{(1)}_{yy}$ and $\chi^{(1)}_{zz}$.

(2) For $\chi^{(2)}$, only eight components are nonzero, corresponding to all possible combinations of x and y indices. Of these, only two are independent. Explicitly, $\chi^{(2)}_{xyy}=\chi^{(2)}_{yxy}=\chi^{(2)}_{yyx}=-\chi^{(2)}_{xxx}$ and $\chi^{(2)}_{yxx}=\chi^{(2)}_{xyx}=\chi^{(2)}_{xxy}=-\chi^{(2)}_{yyy}$.

(3) For $\chi^{(3)}$, there are 41 nonzero elements, corresponding to all the elements in which the z index appears in pairs. Of these, 19 are independent. Explicitly, $\chi^{(3)}_{xxyy}=\chi^{(3)}_{yyxx}$, $\chi^{(3)}_{xyyx}=\chi^{(3)}_{yxxy}$, and $\chi^{(3)}_{xyxy}=\chi^{(3)}_{yxyx}$, while $\chi^{(3)}_{yyxy}=-\chi^{(3)}_{xxyx}$, $\chi^{(3)}_{yxyy}=-\chi^{(3)}_{xyxx}$, and $\chi^{(3)}_{xyyy}=-\chi^{(3)}_{yxxx}$. Also, $\chi^{(3)}_{xxxx}=\chi^{(3)}_{xxyy}+\chi^{(3)}_{xyyx}+\chi^{(3)}_{xyxy}=\chi^{(3)}_{yyyy}$ and $\chi^{(3)}_{xxxy}=\chi^{(3)}_{yyxy}+\chi^{(3)}_{yxyy}+\chi^{(3)}_{xyyy}=-\chi^{(3)}_{yyyx}$. Components with z indices are omitted since they have no active role in the calculations.

Finally, we allow for a small birefringence in the transverse dimensions by assuming that $|k_x-k_y|\lessdot\epsilon$, due to a slight effective difference between $\chi^{(1)}_{xx}$ and $\chi^{(1)}_{yy}$ which could be obtained in various ways, e.g., imperfections of the crystal or a waveguide, and would break the degeneracy of the propagation modes. Of course, in a waveguide we would neglect one of the transverse derivatives. The presence of artificial birefringence allows us to illustrate some of the scenarios that can appear when the derivation is performed for different materials. As we will see, the asymptotically small size of the birefringence terms plays an important role in determining the precise details of the resulting NLSM equations.

A. Derivation of the vector equations

The electric field is expanded as in Eq. (2.6). However, in this case we include both transverse components at $O(\epsilon)$, i.e., we take $E^{(1)}_{x,\pm1}\neq0$, $E^{(1)}_{y,\pm1}\neq0$, and $E^{(1)}_{x,\pm1}=0$. The perturbation expansion is similar to the case discussed above. The main difference from the scalar case is the presence of two different phases for the x and y components; namely, the (fast) variable θ defined in (2.4a) is substituted by θ_x and θ_y, with

$$\theta_j:=k_jz-\omega t, \qquad (4.1)$$

where each of the wave numbers $k_j(\omega)$ is to be determined in the following. Also, the definition of the (slow) retarded time is now modified to be $T=\epsilon(t-z/\bar{v})$, where the mean group velocity $\bar{v}$ is also to be determined later. As a consequence, some care must be taken in replacing the z derivative with the proper slow and fast counterparts [Eq. (2.5c)], since each transverse component now evolves with its own characteristic wave number. Therefore we expand the electric field as

$$E^{(n)}_j=\sum_{m=-n}^{n}e^{im\theta_j}E^{(n)}_{j,m}(X,Y,Z,T). \qquad (4.2)$$

[We should point out that, as long as $E^{(n)}_{z,m}$ is slowly varying, the choice of θ_z is arbitrary, since the z component of the electric field is only driven by the x and y components, as in the scalar case. In other words, E_z will turn out to be independent of θ_z; cf. Eq. (4.8) below.] In turn, Eq. (2.11) needs to be replaced by

$$L_{j;m}=\left(imk_j-\frac{\epsilon}{\bar{v}}\partial_T+\epsilon^2\partial_Z^2\right)^2+\epsilon^2(\partial_X^2+\partial_Y^2)+\kappa_j^2(\omega_m+i\epsilon\partial_T). \qquad (4.3)$$

Equations (2.15) change accordingly. In spite of these modifications, and although the calculations are considerably more involved, the analysis proceeds almost exactly as in the scalar case. In what follows we only concentrate on the differences between the two cases.

At $O(\epsilon)$ we find the respective wave numbers as a function of ω. As it may be expected, k_x and k_y are given by $k_j=\kappa_j(\omega)$, where $\kappa_j(\omega)$ is still defined by (2.12). At $O(\epsilon^2)$, for the fundamental in each transverse component $a=x,y$ we find

$$2ik_a\left(k'_a-\frac{1}{\bar{v}}\right)\frac{\partial}{\partial T}E^{(1)}_{a,1}=0. \qquad (4.4)$$

The respective group velocities are given by $v_j = 1/k'_j$, as before. However, due to birefringence we have $v_x \neq v_y$. The mean velocity is then defined as $\overline{v}(\omega) = 1/\overline{k'}(\omega)$, where

$$\overline{k'}(\omega) = \tfrac{1}{2}(k'_x(\omega) + k'_y(\omega)). \tag{4.5}$$

Then, if $k'_x(\omega) - k'_y(\omega) \sim O(\epsilon)$, it is useful to introduce the $O(1)$ temporal walk-off coefficients

$$w_a = 2k_a(k'_a(\omega) - \overline{k'}(\omega))/\epsilon. \tag{4.6}$$

In fact, since the difference $k'_a - \overline{k'}$ is $O(\epsilon)$, the $O(\epsilon^2)$ equations at the fundamental become effectively residuals of $O(\epsilon^3)$, which are carried at next order in the expansion.

Also, at $O(\epsilon^2)$ for $m=2$, we obtain the second harmonic components in the transverse coordinates as

$$E^{(2)}_{a,2} = \frac{1}{\Delta_a^2(\omega)}[\chi_{aaa}(\omega,\omega)(E^{(1)}_{a,1})^2 - \chi_{a\overline{aa}}(\omega,\omega)\times(E^{(1)}_{\overline{a},1})^2 e^{2i(\theta_{\overline{a}}-\theta_a)} - 2\chi_{\overline{aa}a}(\omega,\omega)E^{(1)}_{a,1}E^{(1)}_{\overline{a},1}e^{i(\theta_a-\theta_{\overline{a}})}], \tag{4.7}$$

where a indicates either x or y, and $\overline{a}$ is the other transverse coordinate [with $\Delta_j(\omega)$ still given by (3.3)]. Note how the more complicated tensor structure of $\chi^{(2)}$ and the presence of a y component at the fundamental generate the appearance of many more coupling combinations in Eq. (4.7) compared to Eq. (3.2). As in the scalar case, at $O(\epsilon^2)$ all other harmonics are found to be zero, except for all the dc fields, which are unknown at this level and need to be determined at higher order in the expansion.

From the divergence law at the fundamental we get the $O(\epsilon^2)$ correction to the fields at the fundamental as

$$E^{(2)}_{z,1} = \frac{in_x^2(\omega)}{k_z(\omega)n_z^2(\omega)}\frac{\partial}{\partial X}E^{(1)}_{x,1}e^{i(\theta_x-\theta_z)} + \frac{in_y^2(\omega)}{k_y(\omega)n_z^2(\omega)}\frac{\partial}{\partial Y}E^{(1)}_{y,1}e^{i(\theta_y-\theta_z)}, \tag{4.8}$$

which is to be compared to Eq. (3.4). Similarly, at $O(\epsilon^3)$, from the divergence law at dc we get the z component of the dc field in terms of transverse fields:

$$\frac{\partial}{\partial T}E^{(2)}_{z,0} = \frac{\overline{v}(\omega)}{n_z^2(0)}\left[n_x^2(0)\frac{\partial}{\partial X}E^{(2)}_{x,0} + n_y^2(0)\frac{\partial}{\partial Y}E^{(2)}_{y,0}\right] + 2\frac{\overline{v}(\omega)}{n_z^2(0)}\left[\hat{\chi}^{(2)}_{xxx}(\omega,-\omega)\frac{\partial}{\partial X} - \hat{\chi}^{(2)}_{yyy}(\omega,-\omega)\frac{\partial}{\partial Y}\right]\times(|E^{(1)}_{x,1}|^2 - |E^{(1)}_{y,1}|^2) - 2\frac{\overline{v}(\omega)}{n_z^2(0)}\left[\hat{\chi}^{(2)}_{xxx}(\omega,-\omega)\frac{\partial}{\partial X} + \hat{\chi}^{(2)}_{yyy}(\omega,-\omega)\frac{\partial}{\partial Y}\right]\mathrm{Re}[E^{(1)}_{x,1}(E^{(1)}_{y,1})^* e^{i(\theta_x-\theta_y)}], \tag{4.9}$$

which is the analog of Eq. (3.6). The difference $|E^{(1)}_{x,1}|^2 - |E^{(1)}_{y,1}|^2$ in Eq. (4.9) arises from the particular tensor structure belonging to the symmetry class considered (namely, the fact that $\chi^{(2)}_{xyy} = -\chi^{(2)}_{xxx}$), and should not be considered as a generic feature of quadratic materials.

From the $O(\epsilon^3)$ contribution of the wave equation at the fundamental we find the evolution of the transverse components of the optical field. Define

$$E^{(1)}_{x,1} =: A_x(X,Y,T,Z), \quad E^{(2)}_{x,0} =: \phi_x(X,Y,T,Z), \tag{4.10a}$$

$$E^{(1)}_{y,1} =: A_y(X,Y,T,Z), \quad E^{(2)}_{y,0} =: \phi_y(X,Y,T,Z). \tag{4.10b}$$

Then A_x, A_y are found to satisfy the following system of coupled NLSM equations:

$$\left[2ik_a\frac{\partial}{\partial Z} + w_a\frac{\partial}{\partial T} + (1-\delta_{ax}\alpha_{a,1})\frac{\partial^2}{\partial X^2} + (1-\delta_{ay}\alpha_{a,1})\frac{\partial^2}{\partial Y^2} - k_a k''_a\frac{\partial^2}{\partial T^2}\right]A_a - \alpha_{\overline{a},1}\frac{\partial^2}{\partial X\partial Y}A_{\overline{a}}e^{-i(\theta_a-\theta_{\overline{a}})} + (\tilde{M}_{a,1}|A_a|^2 + \tilde{M}_{a,2}|A_{\overline{a}}|^2)A_a + \tilde{M}_{a,3}A_{\overline{a}}^2A_a^* e^{-2i(\theta_a-\theta_{\overline{a}})} + (\tilde{M}_{a,4}|A_a|^2 + \tilde{M}_{a,5}|A_{\overline{a}}|^2)A_{\overline{a}}e^{-i(\theta_a-\theta_{\overline{a}})} + \tilde{M}_{a,6}A_{\overline{a}}^2A_{\overline{a}}^* e^{i(\theta_a-\theta_{\overline{a}})} + (M_{a,0}\phi_a - M_{\overline{a},0}\phi_{\overline{a}})A_a - (M_{a,0}\phi_a + M_{\overline{a},0}\phi_{\overline{a}})A_{\overline{a}}e^{-i(\theta_a-\theta_{\overline{a}})} = 0, \tag{4.11a}$$

where again a denotes either x or y, and $\overline{a}$ is the other transverse coordinate. Note that Eqs. (4.11a) are characterized by the presence of additional nonlinear combinations compared to Eq. (3.14a). In particular, note the direct coupling between the two optical fields *and* the coupling of either optical field to the dc field in the other transverse coordinate. As in the scalar case the evolution of the mean fields ϕ_x, ϕ_y is captured at $O(\epsilon^4)$:

$$\left[(1-\delta_{ax}\alpha_{a,0})\frac{\partial^2}{\partial X^2} + (1-\delta_{ay}\alpha_{a,0})\frac{\partial^2}{\partial Y^2} + s_a\frac{\partial^2}{\partial T^2}\right]\phi_a - \alpha_{\overline{a},0}\frac{\partial^2}{\partial X\partial Y}\phi_{\overline{a}} - \left[N_{a,1}\frac{\partial^2}{\partial T^2} - \delta_{ax}N_{a,2}\frac{\partial^2}{\partial X^2} - \delta_{ay}N_{a,2}\frac{\partial^2}{\partial Y^2} + N_{\overline{a},2}\frac{\partial^2}{\partial X\partial Y}\right](|A_a|^2 - |A_{\overline{a}}|^2) + \left[N_{\overline{a},1}\frac{\partial^2}{\partial T^2} - \delta_{ax}N_{\overline{a},2}\frac{\partial^2}{\partial X^2} - \delta_{ay}N_{\overline{a},2}\frac{\partial^2}{\partial Y^2} - N_{a,2}\frac{\partial^2}{\partial X\partial Y}\right]\mathrm{Re}(A_a A_{\overline{a}}^* e^{i(\theta_a-\theta_{\overline{a}})}) = 0. \tag{4.11b}$$

Again, we note the presence of many more nonlinear forcing terms in Eqs. (4.11b) as opposed to Eqs. (3.14b) and (3.14c). As in the scalar NLSM case discussed before, some of these combinations result from the contribution of the divergence of the electric field, $\nabla \cdot \mathbf{E}$, in Eq. (2.1), through $\alpha_{j,m}$, and would be missed by using a scalar wave equation instead of the full vector Eq. (2.1).

The coefficients $\alpha_{a,m}, M_{a,m}, N_{a,m}$ are still defined by Eqs. (3.10), while

$$s_a = \frac{1}{\overline{v}^2(\omega)} - \frac{1}{c_a^2(0)}, \tag{4.12a}$$

δ_{ij} is the Kronecker symbol ($\delta_{ij}=1$ if $i=j$, $\delta_{ij}=0$ if $i \neq j$), and the coefficients $\tilde{M}_{a,1},\ldots,\tilde{M}_{a,6}$ are

$$\tilde{M}_{a,1} = \frac{\omega^2}{c^2}(3b_{a,1}+2c_{a,1}), \quad \tilde{M}_{a,2} = \frac{\omega^2}{c^2}(2b_{a,1}+4c_{\bar{a},2}),$$

$$\tilde{M}_{a,3} = \frac{\omega^2}{c^2}(b_{a,1}-2c_{a,1}), \tag{4.12b}$$

$$\tilde{M}_{a,4} = \frac{\omega^2}{c^2}(2b_{a,2}+4c_{\bar{a},3}), \quad \tilde{M}_{a,5} = \frac{\omega^2}{c^2}(3b_{a,2}+2c_{a,4}),$$

$$\tilde{M}_{a,6} = \frac{\omega^2}{c^2}(b_{a,2}-2c_{a,4}), \tag{4.12c}$$

where

$$b_{a,1} = \hat{\chi}^{(3)}_{aaaa}(\omega,\omega,-\omega), \quad b_{a,2} = \hat{\chi}^{(3)}_{\overline{aaaa}}(\omega,\omega,-\omega) \tag{4.12d}$$

and

$$c_{a,1} = \hat{\chi}^{(2)}_{aaa}(2\omega,-\omega)\hat{\chi}^{(2)}_{aaa}(\omega,\omega)/\Delta_a^2(\omega) + \hat{\chi}^{(2)}_{\overline{aaa}}(2\omega,-\omega)\hat{\chi}^{(2)}_{\overline{aaa}}(\omega,\omega)/\Delta_{\bar{a}}^2(\omega), \tag{4.12e}$$

$$c_{a,2} = \hat{\chi}^{(2)}_{aaa}(2\omega,-\omega)\hat{\chi}^{(2)}_{aaa}(\omega,\omega)/\Delta_{\bar{a}}^2(\omega) + \hat{\chi}^{(2)}_{\overline{aaa}}(2\omega,-\omega)\hat{\chi}_{\overline{aa}a}(2)(\omega,\omega)/\Delta_a^2(\omega), \tag{4.12f}$$

$$c_{a,3} = \hat{\chi}^{(2)}_{aaa}(2\omega,-\omega)\hat{\chi}^{(2)}_{\overline{aaa}}(\omega,\omega)/\Delta_a^2(\omega) - \hat{\chi}^{(2)}_{\overline{aaa}}(2\omega,-\omega)\hat{\chi}^{(2)}_{aaa}(\omega,\omega)/\Delta_{\bar{a}}^2(\omega), \tag{4.12g}$$

$$c_{a,4} = \hat{\chi}^{(2)}_{\overline{aaa}}(2\omega,-\omega)\hat{\chi}^{(2)}_{aaa}(\omega,\omega)/\Delta_a^2(\omega) - \hat{\chi}^{(2)}_{aaa}(2\omega,-\omega)\hat{\chi}^{(2)}_{\overline{aaa}}(\omega,\omega)/\Delta_{\bar{a}}^2(\omega), \tag{4.12h}$$

with $\Delta_j(\omega)$ given by (3.3). The particular symmetry class considered ensures that a symmetrical result is obtained for x and y components, and that the couplings due to $\chi^{(3)}$ contributions are written as a simple extension of those valid for isotropic materials.

B. Reductions and physical limits

Together, the coupled NLSM equations (4.11) constitute the fundamental system that governs the evolution of the electromagnetic pulse in the material considered. They generalize Eqs. (3.14) in the case where the incident field has nonzero components along both transversal axes of the material. As far as we know, the vector system of $(3+1)$-dimensional NLSM equations (4.11) has no known counterpart in other physical situations.

As a result of the perturbation expansion, the electric field $\mathbf{E}(x,y,z,t)$ has the following decomposition:

$$\begin{aligned} E_a(x,y,z,t) = {} & \epsilon[E_{a,1}^{(1)}(X,Y,Z,T)e^{i\theta_a} + E_{a,1}^{(1)}(X,Y,Z,T)^* e^{-i\theta_a}] \\ & + \epsilon^2[E_{a,2}^{(2)}(X,Y,Z,T)e^{2i\theta_a} + E_{a,2}^{(2)} \\ & \times (X,Y,Z,T)^* e^{-2i\theta_a} + E_{a,0}^{(2)}(X,Y,Z,T)] \\ & + O(\epsilon^3), \end{aligned} \tag{4.13a}$$

$$\begin{aligned} E_z(x,y,z,t) = {} & \epsilon^2[E_{z,1}^{(1)}(X,Y,Z,T)e^{i\theta_z} + E_{z,1}^{(1)}(X,Y,Z,T)^* e^{-i\theta_z} \\ & + E_{z,0}^{(2)}(X,Y,Z,T)] + O(\epsilon^3), \end{aligned} \tag{4.13b}$$

with $a = x,y$, where $E_{x,2}$ is given by Eq. (4.7), $E_{z,1}$ is given by Eq. (4.8), $E_{z,0}$ is given by Eq. (4.9), and $E_{a,1}$ and $E_{a,0}$ are determined by the vector NLSM equations (4.11). Equations (4.13) are to be compared to Eqs. (3.11), which are valid in the scalar case.

It is interesting to note that the usual coupled NLS system is a limiting reduction of (4.11). Namely, in the case where $\chi^{(2)}=0$, isotropic materials, we have $M_{a,0}=N_{a,1}=N_{a,2}=\phi_a=0$. Hence Eqs. (4.11) reduce to the well-known coupled NLS equations (cf. Ref. [30]):

$$\begin{aligned} & \left[2ik_a\frac{\partial}{\partial Z} + w_a\frac{\partial}{\partial T} + (1-\delta_{ax}\alpha_{a,1})\frac{\partial^2}{\partial X^2} \right. \\ & \quad \left. + (1-\delta_{ay}\alpha_{a,1})\frac{\partial^2}{\partial Y^2} - k_a k_a''\frac{\partial^2}{\partial T^2}\right]A_a \\ & \quad - \alpha_{\bar{a},1}\frac{\partial^2}{\partial X\partial Y}A_{\bar{a}}e^{-i(\theta_a-\theta_{\bar{a}})} + \tilde{M}_{a,3}(3|A_a|^2+2|A_{\bar{a}}|^2)A_a \\ & \quad + \tilde{M}_{a,3}A_{\bar{a}}^2A_a^* e^{-2i(\theta_a-\theta_{\bar{a}})} + \tilde{M}_{a,6}(2|A_a|^2 \\ & \quad + 3|A_{\bar{a}}|^2)A_{\bar{a}}e^{-i(\theta_a-\theta_{\bar{a}})} + \tilde{M}_{a,6}A_a^2A_{\bar{a}}^* e^{i(\theta_a-\theta_{\bar{a}})} = 0, \end{aligned} \tag{4.14}$$

where $\tilde{M}_{a,1} = \frac{3}{2}\tilde{M}_{a,2} = 3\tilde{M}_{a,3} = (\omega^2/c^2)\hat{\chi}^{(3)}_{aaaa}(\omega,\omega,-\omega)$ and $\tilde{M}_{a,5} = \frac{3}{2}\tilde{M}_{a,4} = 3\tilde{M}_{a,6} = (\omega^2/c^2)\hat{\chi}^{(3)}_{\overline{aaaa}}(\omega,\omega,-\omega)$. Also, the vector Eqs. (4.11) reduce to the scalar system (3.14) if $A_y = \chi^{(2)}_{yyy} = 0$.

In the more general case where $\chi^{(2)} \neq 0$, a number of different physical situations occur according to the magnitude of the term $\Delta k = k_x - k_y$. We analyze each of them separately.

Case 1. Consider first the case in which $|k_x - k_y|$ is nonzero and $O(\epsilon)$. Introducing $k_x - k_y \equiv \epsilon k_0$, we can write the phase difference as $\theta_x - \theta_y = k_0 Z/\epsilon$. Therefore we see that all the terms which contain phase difference $\theta_x - \theta_y$ are rapidly varying in Z, and will not contribute to the system when integrated over a distance $Z\sim O(1)$. Equations (4.11) then become

$$\left[2ik_a\frac{\partial}{\partial Z}+w_a\frac{\partial}{\partial T}+(1-\delta_{ax}\alpha_{a,1})\,\frac{\partial^2}{\partial X^2}+(1-\delta_{ay}\alpha_{a,1})\,\frac{\partial^2}{\partial Y^2}\right.$$
$$\left.-k_ak_a''\frac{\partial^{2\prime}}{\partial T^2}\right]A_a+[\tilde M_{a,1}|A_a|^2+\tilde M_{a,2}|A_{\bar a}|^2+M_{a,0}\phi_a$$
$$-M_{\bar a,0}\phi_{\bar a}]A_a=0, \tag{4.15a}$$

$$\left[(1-\delta_{ax}\alpha_{a,0})\,\frac{\partial^2}{\partial X^2}+(1-\delta_{ay}\alpha_{a,0})\,\frac{\partial^2}{\partial Y^2}+s_a\frac{\partial^2}{\partial T^2}\right]\phi_a$$
$$-\alpha_{\bar a,0}\frac{\partial^2}{\partial X\partial Y}\phi_{\bar a}-\left[N_{a,1}\frac{\partial^2}{\partial T^2}-\delta_{ax}N_{a,2}\frac{\partial^2}{\partial X^2}-\delta_{ay}N_{a,2}\frac{\partial^2}{\partial Y^2}\right.$$
$$\left.+N_{\bar a,2}\frac{\partial^2}{\partial X\partial Y}\right](|A_a|^2-|A_{\bar a}|^2)=0, \tag{4.15b}$$

where the temporal walk-off coefficient w_a is now $O(1)$. In this case there is no coupling to ϕ_y in the equation for A_x (nor to ϕ_x in the equation for A_y). However, the equations for the dc fields are still driven by *both* optical fields, in contrast to the scalar case. Thus, Eqs. (4.15) constitute a nonlinearly coupled system.

Case 2. The opposite situation emerges when there is no phase mismatch between the x and y components of the electric field, that is, if $k_x=k_y$. In this case all the exponentials in (4.11) become unity, and the equations are

$$\left[2ik_a\frac{\partial}{\partial Z}+(1-\delta_{ax}\alpha_{a,1})\,\frac{\partial^2}{\partial X^2}+(1-\delta_{ay}\alpha_{a,1})\,\frac{\partial^2}{\partial Y^2}\right.$$
$$\left.-k_ak_a''\frac{\partial^2}{\partial T^2}\right]A_a-\alpha_{\bar a,1}\frac{\partial^2}{\partial X\partial Y}A_{\bar a}+(\tilde M_{a,1}|A_a|^2$$
$$+\tilde M_{a,2}|A_{\bar a}|^2)A_a+\tilde M_{a,3}A_{\bar a}^2A_a^*+(\tilde M_{a,4}|A_a|^2$$
$$+\tilde M_{a,5}|A_{\bar a}|^2)A_{\bar a}+\tilde M_{a,6}A_a^2A_{\bar a}^*+(M_{a,0}\phi_a-M_{\bar a,0}\phi_{\bar a})A_a$$
$$-(M_{a,0}\phi_a+M_{\bar a,0}\phi_{\bar a})A_{\bar a}=0, \tag{4.16a}$$

$$\left[(1-\delta_{ax}\alpha_{a,0})\,\frac{\partial^2}{\partial X^2}+(1-\delta_{ay}\alpha_{a,0})\,\frac{\partial^2}{\partial Y^2}+s_a\frac{\partial^2}{\partial T^2}\right]\phi_a$$
$$-\alpha_{\bar a,0}\frac{\partial^2}{\partial X\partial Y}\phi_{\bar a}-\left[N_{a,1}\frac{\partial^2}{\partial T^2}-\delta_{ax}N_{a,2}\frac{\partial^2}{\partial X^2}-\delta_{ay}N_{a,2}\frac{\partial^2}{\partial Y^2}\right.$$
$$\left.+N_{\bar a,2}\frac{\partial^2}{\partial X\partial Y}\right](|A_a|^2-|A_{\bar a}|^2)+\left[N_{\bar a,1}\frac{\partial^2}{\partial T^2}-\delta_{ax}N_{\bar a,2}\frac{\partial^2}{\partial X^2}\right.$$
$$\left.-\delta_{ay}N_{\bar a,2}\frac{\partial^2}{\partial Y^2}-N_{a,2}\frac{\partial^2}{\partial X\partial Y}\right]\mathrm{Re}(A_aA_{\bar a}^*)=0. \tag{4.16b}$$

Of course, in this case there is no temporal walk-off term, i.e., $w_a=0$.

Case 3. Finally, if the difference between k_x and k_y is nonzero and $O(\epsilon^2)$, i.e., if $k_x-k_y\equiv\epsilon^2k_0$, then $\theta_x-\theta_y=k_0Z$, and the dynamics of the systems is effectively described by the full system of Eqs. (4.11), where now however w_a is $O(\epsilon)$ and can be ignored:

$$\left[2ik_a\frac{\partial}{\partial Z}+(1-\delta_{ax}\alpha_{a,1})\,\frac{\partial^2}{\partial X^2}+(1-\delta_{ay}\alpha_{a,1})\,\frac{\partial^2}{\partial Y^2}\right.$$
$$\left.-k_ak_a''\frac{\partial^2}{\partial T^2}\right]A_a-\alpha_{\bar a,1}\frac{\partial^2}{\partial X\partial Y}A_{\bar a}e^{-i(\theta_a-\theta_{\bar a})}+(\tilde M_{a,1}|A_a|^2$$
$$+\tilde M_{a,2}|A_{\bar a}|^2)A_a+\tilde M_{a,3}A_{\bar a}^2A_a^*e^{-2i(\theta_a-\theta_{\bar a})}+(\tilde M_{a,4}|A_a|^2$$
$$+\tilde M_{a,5}|A_{\bar a}|^2)A_{\bar a}e^{-i(\theta_a-\theta_{\bar a})}$$
$$+\tilde M_{a,6}A_a^2A_{\bar a}^*e^{i(\theta_a-\theta_{\bar a})}+(M_{a,0}\phi_a-M_{\bar a,0}\phi_{\bar a})A_a$$
$$-(M_{a,0}\phi_{\bar a}-M_{\bar a,0}\phi_{\bar a})A_{\bar a}e^{-i(\theta_a-\theta_{\bar a})}=0, \tag{4.17a}$$

$$\left[(1-\delta_{ax}\alpha_{a,0})\,\frac{\partial^2}{\partial X^2}+(1-\delta_{ay}\alpha_{a,0})\,\frac{\partial^2}{\partial Y^2}+s_a\frac{\partial^2}{\partial T^2}\right]\phi_a$$
$$-\alpha_{\bar a,0}\frac{\partial^2}{\partial X\partial Y}\phi_{\bar a}-\left[N_{a,1}\frac{\partial^2}{\partial T^2}-\delta_{ax}N_{a,2}\frac{\partial^2}{\partial X^2}-\delta_{ay}N_{a,2}\frac{\partial^2}{\partial Y^2}\right.$$
$$\left.+N_{\bar a,2}\frac{\partial^2}{\partial X\partial Y}\right](|A_a|^2-|A_{\bar a}|^2)+\left[N_{\bar a,1}\frac{\partial^2}{\partial T^2}-\delta_{ax}N_{\bar a,2}\frac{\partial^2}{\partial X^2}\right.$$
$$\left.-\delta_{ay}N_{\bar a,2}\frac{\partial^2}{\partial Y^2}-N_{a,2}\frac{\partial^2}{\partial X\partial Y}\right]\mathrm{Re}(A_aA_{\bar a}^*e^{i(\theta_a-\theta_{\bar a})})=0. \tag{4.17b}$$

It is important to stress that, if $\Delta k=k_x-k_y$ is not $O(\epsilon)$ or smaller, multiple time and space scales are still present in Eqs. (4.11). In this particular case Eqs. (4.11) are not in a sense true asymptotic equations, and further analysis is necessary in order to extract the true asymptotic behavior of the system. More precisely, if $|k_x-k_y|\gg\epsilon$, equations such as the ones discussed here cannot result. For example, if $|k_x-k_y|\sim O(1)$, it follows that $|v_x-v_y|\sim O(1)$ and $w_x,w_y\sim O(1/\epsilon)$, which implies that the walk-off terms cannot be carried to $O(\epsilon^3)$. The system is then governed by different equations, in general valid over different space-time scales. Therefore the condition $|k_x-k_y|\lesssim O(\epsilon)$ poses a limitation on the physical situations that can be described via vector NLSM equations such as the ones presented in this paper.

As for the properties of Eqs. (4.11), (4.15), and (4.16), we expect some of the observations made in the scalar case to carry through. Nonetheless, an adequate study of the equations and their behavior, as well as a characterization of the solutions, is essentially an open problem.

V. SPECIAL SOLUTIONS

In this section we consider the scalar NLSM equations derived in Sec. III and we show that stable localized multidimensional pulses exist which are driven by appropriate mean fields, even in the more general nonintegrable case.

A. Nondimensionalization and rescalings

If the Y dependence of A and ϕ can be neglected, or if it is already taken into account when dealing with the linear modes (e.g., in a waveguide configuration), Eqs. (3.8) become effectively $(2+1)$-dimensional. In this case we introduce nondimensional variables and fields as

$$\tau = T/T_0, \quad \xi = X/X_0, \quad \zeta = Z/Z_0,$$

$$q = A/A_0, \quad Q = \phi/\phi_0, \tag{5.1}$$

with

$$T_0 = [|k_x''|Z_0/2]^{1/2}, \quad X_0 = [r_{x,0}/|s_x|]^{1/2} T_0, \tag{5.2a}$$

$$\phi_0 = \frac{2k_x}{Z_0 M_{x,0}}, \quad A_0^2 = \frac{2k_x}{Z_0 M_{x,0} N}, \tag{5.2b}$$

where Z_0 and N are for now arbitrary, and where for convenience we introduced

$$r_{j,m} = \frac{n_j^2(m\omega)}{n_z^2(m\omega)} = 1 - \alpha_{j,m}. \tag{5.2c}$$

In this way we can write the optical NLSM equations for a tetragonal 4 mm material in the case of anomalous dispersion $(k''<0)$ as the following dimensionless system:

$$iq_\zeta + a_1 q_{\xi\xi} + q_{\tau\tau} + (a_2|q|^2 + Q)q = 0, \tag{5.3a}$$

$$Q_{\xi\xi} - Q_{\tau\tau} = c_1(|q|^2)_{\tau\tau} - c_2(|q|^2)_{\xi\xi}, \tag{5.3b}$$

where

$$a_1 = \frac{r_{x,1}}{r_{x,0}} \frac{|s_x|}{k_x|k_x''|}, \quad a_2 = \frac{M_{x,1}}{M_{x,0}N}, \tag{5.4a}$$

$$c_1 = \frac{N_{x,1}}{|s_x|N}, \quad c_2 = \frac{N_{x,2}}{r_{x,0}N}, \tag{5.4b}$$

with $N_{x,1}$ and $N_{x,2}$ given by Eqs. (3.10), and where the subscripts ξ, ζ, and τ on q and Q denote partial derivatives. Note that the integrable case presented in Eq. (3.17) can also be rewritten in terms of Eq. (5.3) by simply identifying $t=\zeta$, $x=\xi$, $y=\tau$, $A=q$, $\Phi_x=-Q$, $\sigma=a_1=a_2=1$, $c_1=0$, and $c_2=2$.

Next we perform a rotation of coordinates to the characteristic frame of reference of Eq. (5.3b):

$$\tau = (\xi' + \tau')/\sqrt{2}, \quad \xi = (\xi' - \tau')/\sqrt{2}. \tag{5.5}$$

Also, we redefine the mean field as $Q' = Q - \frac{1}{2}(c_1 + c_2)|q|^2$, and we choose the normalization constant N as $N = N_-$, where

$$2N_\pm = \frac{N_{x,1}}{|s_x|} \pm \frac{N_{x,2}}{r_{x,0}} \tag{5.6}$$

(N_+ is defined here for later use). Omitting primes for simplicity, Eqs. (5.3) in the new frame of reference take on a particularly simple form:

$$iq_\zeta + (1-\theta_1)(q_{\xi\xi} + q_{\tau\tau}) + 2\theta_1 q_{\epsilon\tau} + (\theta_2|q|^2 + Q)q = 0, \tag{5.7a}$$

$$2Q_{\xi\tau} = (|q|^2)_{\xi\xi} + (|q|^2)_{\tau\tau}, \tag{5.7b}$$

where the parameters $\theta_{1,2}$ are given by $\theta_1 = \frac{1}{2}(1-a_1)$ and $\theta_2 = a_2 - \frac{1}{2}(c_1 + c_2)$. In Eqs. (5.7) all the properties of the material are absorbed into the values of the constants θ_1 and θ_2, which are explicitly given by

$$\theta_1 = \frac{1}{2}\left(1 - \frac{r_{x,1}|s_x|}{r_{x,0}k_x|k_x''|}\right), \tag{5.8a}$$

$$\theta_2 = \frac{1}{N_-}\left(\frac{M_{x,1}}{M_{x,0}} - N_+\right). \tag{5.8b}$$

The integrable case corresponds to $\theta_1 = \theta_2 = 0$. Equation (5.7b) can be readily solved by expressing the dc field as $Q = U + V$, with

$$U(\xi,\tau,\zeta) = \frac{1}{2}\int_{\tau_*}^{\tau} (|q|^2)_\xi \, d\tau + u_*(\xi,\zeta), \quad V(\xi,\tau,\zeta) = \frac{1}{2}\int_{\xi_*}^{\xi} (|q|^2)_\tau \, d\xi + v_*(\tau,\zeta), \tag{5.9}$$

and where $u_*(\xi,\zeta), v_*(\tau,\zeta)$ are two arbitrary integration constants. If $(\xi_*,\tau_*) = (-\infty,-\infty)$, the corresponding two functions, which we call u_∞ and v_∞, assume the role of boundary conditions of the dc field $Q = U + V$. An alternative but equivalent choice which we will use in the following is $(\xi_*,\tau_*) = (0,0)$, corresponding to functions u_0 and v_0. It is clear that there is a one-to-one correspondence between any choice for u_∞ and v_∞ and any choice for u_0 and v_0. As demonstrated below, the functions u_0, v_0 or u_∞, v_∞ play a key role in the dynamics of the pulse.

B. Special solutions: Integrable case

In the integrable case (that is, when $\theta_1 = \theta_2 = 0$), stable localized multidimensional pulses, often called ''dromions,''are known to exist, which are driven by the mean field through a proper choice of boundary conditions. Explicitly, the one-dromion solution of Eq. (5.7) with $\theta_1 = \theta_2 = 0$ is given by [16,17]

$$q(x_1,x_2,t) = G(x_1,x_2,t)/F(x_1,x_2,t), \tag{5.10}$$

where, for convenience, we set $x_1 = \xi$, $x_2 = \tau$, and $t = \zeta$, and where

$$G(x_1,x_2,t) = \rho e^{\eta_1 + \eta_2 + i(\phi_1 + \phi_2)}, \tag{5.11a}$$

$$F(x_1,x_2,t) = 1 + e^{2\eta_1} + e^{2\eta_2} + \gamma e^{2(\eta_1 + n_2)}, \tag{5.11b}$$

with

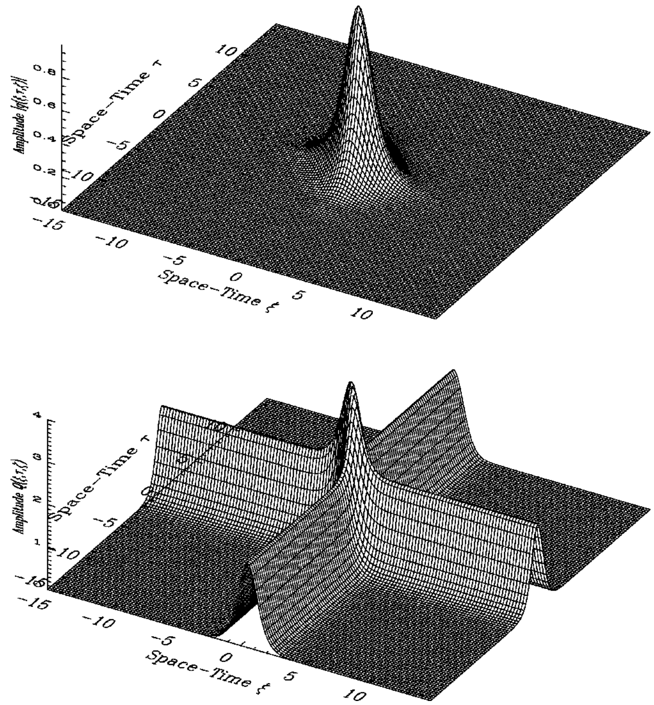

FIG. 1. The stationary dromion solution of the integrable case $(\theta_1=\theta_2=0)$: (a) The optical pulse $|q(\xi,\tau,\zeta)|$; (b) the dc field $Q(\xi,\tau)=U(\xi,\tau)+V(\xi,\tau)$ associated with q.

$$\eta_j(x_j,t)=k_j(x_j-x_{0,j}-2\omega_j t), \tag{5.12a}$$

$$\phi_j(x_j,t)=\omega_j(x_j-x_{0,j})-(\omega_j^2-k_j^2)t, \tag{5.12b}$$

$j=1,2$, with $\rho=2\sqrt{2(\gamma-1)k_1k_2}$, and with $k_j,\omega_j,x_{0,j}$ arbitrary real parameters. The constant γ determines the overall amplitude, the parameters k_j represent the width of the pulse in each respective direction, and the ω_j represent the Cartesian components of the velocity, while the $x_{0,j}$ determine the dromion position. The ''potentials'' $U(x_1,x_2,t)$ and $V(x_1,x_2,t)$ are obtained by integrating Eqs. (5.9) subject to the boundary conditions

$$u_\infty(x_1,t)=2k_1^2\,\mathrm{sech}^2\,\eta_1,\quad v_\infty(x_2,t)=2k_2^2\,\mathrm{sech}^2\,\eta_2. \tag{5.13}$$

The resulting expressions are

$$U(x_1,x_2,t)=2(\ln F(x_1,x_2,t))_{x_1x_2},$$

$$V(x_1,x_2,t)=2(\ln F(x_1,x_2,t))_{x_2x_2}. \tag{5.14}$$

Figure 1(a) shows a typical (stationary) dromion solution, corresponding to $k_1=k_2=1$, $\omega_1=\omega_2=0$, and $\gamma=9$. The corresponding dc field $Q=U+V$ is shown in Fig. 1(b) [note the nonzero boundary conditions $u_\infty(x_1)$ and $v_\infty(x_2)$ corresponding to Eq. (5.13)]. The pulse is located at the intersection of U and V.

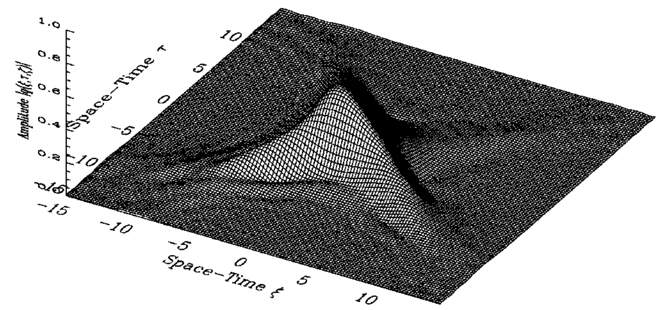

FIG. 2. The output pulse at $\zeta_{\rm out}=2$, for $\theta_1=\theta_2=0.4$, with same initial condition as in Fig. 1(a) but with zero boundary conditions.

C. Special solutions: General case

When θ_1 and/or θ_2 are not zero, to date no localized analytical solutions have been found, and one must resort to numerical simulations. We integrated Eq. (5.7a) with a two-dimensional second order split-step method and Eqs. (5.9) with a second order numerical quadrature routine, for a number of different values of θ_1 and θ_2. Figure 1(a) shows a typical stationary pulse in the integrable case $\theta_1=\theta_2=0$, while Fig. 1(b) represents the corresponding dc field $Q=U+V$ obtained from Eq. (5.9) with u_∞, v_∞ given by Eq. (5.13). If the pulse shown in Fig. 1(a) is used as initial condition for $\theta_1,\theta_2\neq 0$, and the same boundary conditions as in Fig. 1(b) are used for the dc fields numerical simulations show that, even though some radiation is produced, a localized pulse similar to the one in the integrable case persists for a long propagation distance. On the other hand, Fig. 2 shows the output produced after just two propagation distances by the same input pulse as in Fig. 1(a) if the boundary conditions u, v for the dc fields are zero (i.e., $u_\infty=v_\infty=0$), for $\theta_1=\theta_2=0.4$. In this case the pulse decays very quickly, and no localized asymptotic state is obtained. (Note that the pulse disperses along directions which are the analog of the Mach lines associated with the propagation of a supersonic disturbance in a classical fluid.) It is therefore clear that, even in the more general situation $\theta_1\neq 0$, $\theta_2\neq 0$, the dc fields can stabilize the optical pulses, which otherwise would disperse away very quickly without the presence of nonzero boundary conditions. Similar results were found for a wide range of values of θ_1 and θ_2.

A further result can be achieved if u_0, v_0 are used instead of u_∞, v_∞. In this case, generalized stationary solutions are found to exist even when θ_1 and θ_2 are significantly different from zero. To find these solutions, we integrated Eqs. (5.7a) and (5.9) with $\theta_1,\theta_2\neq 0$, using the dromion solution of the integrable case as initial condition and inserting absorbing boundaries at the edges of the two-dimensional grid to remove the radiation shed by the pulse [31]. After the pulse has reached an asymptotic state, we removed the absorbing boundaries and let the pulse evolve according to the NLSM equations, to verify that we have obtained a stationary solution. As an example, in Fig. 3(a) we show the stationary pulse corresponding to $\theta_1=\theta_2=0.4$, while Fig. 3(b) is relative to the case $\theta_1=0$, $\theta_2=-1$. (In particular, this last case implies that, even for defocusing self-interaction, the presence of nonzero asymptotic mean fields is sufficient to main-

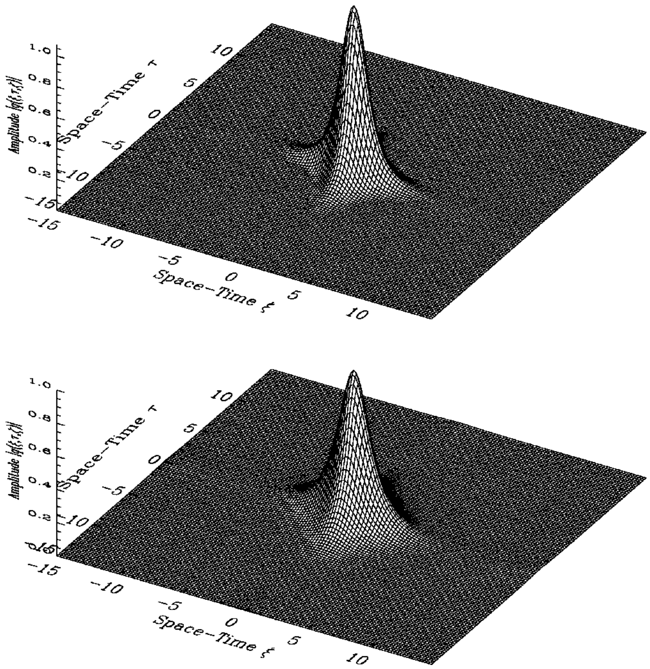

FIG. 3. Stationary solutions of the optical NLSM equations in the nonintegrable case: (a) $\theta_1=\theta_2=0.4$; (b) $\theta_1=0$, $\theta_2=-1$.

tain a localized state.) These findings suggest that stable localized multidimensional pulses are not unique to integrable systems; rather, they are a generic feature of forced evolution equations of this type.

When θ_1 and θ_2 are nonzero, the stationary pulses differ significantly from the corresponding solution of the integrable case. In particular, a numerical study of the equations reveals that θ_1 affects the shape of the pulse, while θ_2 controls its amplitude. More precisely, we find that, starting from a fixed initial condition, for increasingly negative values of θ_2 (implying strong defocusing), the amplitude of the asymptotic stationary state decreases, while for increasing positive values of θ_2 (implying focusing) the final amplitude of the solution increases until, for large enough values of θ_2, the pulse does not asymptote to a stationary state anymore, and, presumably, higher order solutions are obtained.

Some of these features can be explained on the basis of simpler one-dimensional models. To this aim, we consider the parametrically forced NLS equation

$$iq_t+(1/2)q_{xx}+(V(x)+\theta|q|^2)q=0. \tag{5.15}$$

As initial condition and forcing potential we take, respectively, $q_0(x)=a\,\mathrm{sech}\,x$ and $V(x)=A\,\mathrm{sech}^2 x$. By comparison with the usual NLS equation it is clear that, if $A+\theta a^2=1$ [that is, if $a=\sqrt{(1-A)/\theta}$], the initial condition corresponds to the profile of the stationary state $q(x,t)=a\,\mathrm{sech}\,x\,e^{it/2}$. This implies that, in the presence of a strong enough forcing potential ($A>1$ in this case), stationary solutions also exist for $\theta<0$; however, the amplitude of these states decreases with θ. For $\theta>0$ the picture is more complicated. In this case, looking for stationary solutions of the form $q(x,t)=f(x)e^{i\lambda t}$, with $f(x)$ real, Eq. (5.15) leads to the following nonlinear eigenvalue problem:

$$f_{xx}+2(V(x)+\theta f^2-\lambda)f=0. \tag{5.16}$$

The number N of discrete eigenvalues λ_j in Eq. (5.16) depends on the potential $V(x)$, but also on the self-phase modulation coefficient θ. If $N>1$, the solution of Eq. (5.15) may be expected to be a superposition of all the corresponding stationary modes. Since each of these modes has its own frequency, the overall pulse can be expected to undergo periodic or quasiperiodic oscillations, which corresponds to what is observed numerically in the two-dimensional system.

VI. CONCLUSIONS

In this paper we have studied the evolution of a single quasi-monochromatic optical pulse in a multidimensional, nonresonant quadratic material. We have seen that, if there are no resonant wave interactions, and under rather general assumptions, the dynamics of the pulse is governed by equations of nonlinear Schrödinger type with coupling to mean (dc) fields (NLSM). In general, if the incident optical field is polarized along one of the principal axes of the material, scalar equations can be expected to apply. These equations are the $(3+1)$-dimensional analog in optics of a similar type of $(2+1)$-dimensional NLSM equations in water waves. If instead the optical pulse has nonzero polarization projections along both transverse axes, more general vector systems of equations are found depending on the particular physical situations considered. As far as we know, these systems have no known counterpart in other physical situations outside of optics. Even in the scalar case, the dynamics of the optical pulse appears to depend in a critical way upon the interactions with the associated dc fields (cf. also Ref. [29]). In particular, for appropriate choices of boundary conditions, stable localized multidimensional pulses can arise even in nonintegrable cases. These findings suggest that stable localized multidimensional pulses are not unique to integrable systems; rather, they are a generic feature of nonlinear evolution equations with forcing terms like those present in Eq. (5.9). In our case, the presence of small applied dc fields [$O(\epsilon^2)$ in the perturbation expansion] can drive much larger optical pulses [$O(\epsilon)$ in the expansion]. Preliminary studies suggest that the above described dynamical configuration might be realized experimentally, given the wide range of values of θ_1 and θ_2 over which stationary propagation occurs. It is expected that, since these values can be adjusted through linear material properties, proper design of the device structure will ensure pulse propagation within the desired regime. This possibility is particularly interesting because such experiments would allow the production of stable localized multidimensional optical pulses whose dynamics can be electrically controlled by modification of the relevant dc fields.

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